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On a Shooting Algorithm for Sturm–Liouville Eigenvalue Problems with Periodic and Semi-periodic Boundary Conditions

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This paper is concerned with the eigenvalues of Sturm–Liouville problems with periodic and semi-periodic boundary conditions to be approximated by a shooting algorithm. The proposed technique is based on the application of the Floquet theory. Convergence analysis and a general guideline to provide starting values for computed eigenvalues are presented. Some numerical results are also reported.

1. INTRODUCTION

In many practical applications related to mathematical physics and engineering science, very often the solutions of Sturm–Liouville (SL) problems are required (cf. [3, 8, 15]). Methods for computing eigenvalues of SL equations have been dealt with by many researchers in both theoretical and numerical aspects (e.g., cf. [2, 4, 5, 14, 17]). From the computational point of view, two methods are commonly used to solve the SL eigenvalue problems. They are the shooting algorithm and the finite difference method.

In a shooting algorithm, the solution of a boundary value problem (BVP) is obtained by solving a set of the related initial value problems (IVPs). Note that a BVP is characterized by the boundary conditions (i.e., the initial and final value of the solutions) imposed at the two boundary points. In contrast, the conditions (i.e., the initial value and the initial slope) of an IVP are specified at only one point. By application of a shooting algorithm, the eigenvalue of a BVP is computed via numerical integration of the associated IVPs with starting approximation. This technique is usually applied in conjunction with Newton's method to seek the corrected eigenvalues. The major advantage of this approach is that numerical methods for IVPs are well developed, and many efficient and reliable computer programs are readily available in mathematical software libraries such as the IMSL and the NAG.

In a finite difference method, the derivatives in the differential equations are replaced by finite differences. Hence, the eigenvalues of a BVP are obtained by solving the resulting algebraic eigenvalue problem.

In this paper, we consider the Sturm–Liouville equation:

$$-(p(t)\,y')' + q(t)\,y = \lambda s(t)\,y, \qquad 0 \leqslant t \leqslant \omega, \qquad (1.1)$$

with the boundary conditions

$$y(0) = \alpha y(\omega), \qquad y'(0) = \alpha y'(\omega). \qquad (1.2)$$

Here $\alpha = 1$ corresponds to periodic and $\alpha = -1$ to semi-periodic boundary conditions, and $p'(t)$, $q(t)$, and $s(t)$ are real-valued, piecewise continuous, and periodic with period ω. The functions $p(t)$ and $s(t)$ are positive in $[0, \omega]$. Let $\{\lambda_i\}$ and $\{\mu_i\}$ be the eigenvalues of the SL equation with periodic and semi-periodic boundary conditions, respectively. It has been proven that the eigenvalues are ordered as follows (cf. [9]):

$$\lambda_0 < \mu_0 \leqslant \mu_1 < \lambda_1 \leqslant \lambda_2 < \mu_2 \leqslant \mu_3 < \lambda_3 \leqslant \lambda_4 < \cdots. \qquad (1.3)$$

The popular method for the periodic or semi-periodic SL eigenvalue problems (1.1)–(1.2) is the finite difference method. However, the solutions of resulting matrix eigenproblems require more work than those for problems with separated boundary conditions (cf. [7, 10]). Moreover, when eigenvalues λ_m's are required for modest m's (e.g., in seismology [16]), one usually has to, even with asymptotic correction technique [1], use a matrix of very high order (cf. [11]). An alternative way is to employ the shooting method. But when this technique is applied to the SL equation, subject to periodic or semi-periodic boundary conditions, not only starting values have to be prescribed, but the initial values are also unknown. In [12], the author exploited the monotone property of phrase function about the eigenvalue parameter to handle this difficulty and consequently extended the well-known Prüfer method to periodic cases. In this paper, we present another approach which is

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based on the Floquet theory of periodic differential equations. In this case, the initial conditions are always fixed (cf. Section 2).

The format of this paper is organized as follows. The development of a shooting algorithm for SL eigenvalue problems with periodic or semi-periodic boundary conditions is described in Section 2. Section 3 presents the convergence analysis and a general guideline to provide starting values. In Section 4, the performance of the shooting algorithm is illustrated by computational results. Conclusions are given in Section 5.

2. FLOQUET THEORY AND A SHOOTING ALGORITHM

Consider the Sturm–Liouville eigenvalue problem with periodic boundary conditions Eqs. (1.1) and (1.2). From Floquet theory (cf. [9], Chap. 1]), it can be shown that there exists a non-zero constant ρ and a non-trivial solution $\psi(t, \lambda)$ such that

$$\psi(t+\omega, \lambda)=\rho\psi(t, \lambda), \tag{2.1}$$

where ω denotes the period. It is clear that $\rho=1$ corresponds to a periodic solution.

Now let $\phi_1(t, \lambda)$ and $\phi_2(t, \lambda)$ be two linearly independent solutions of the SL problem which satisfy the following initial conditions:

$$\begin{aligned} &\phi_1(0, \lambda)=1, \qquad \phi_1'(0, \lambda)=0, \\ &\phi_2(0, \lambda)=0, \qquad \phi_2'(0, \lambda)=1. \end{aligned} \tag{2.2}$$

Using the standard definition of the Wronskian, it gives

$$W(\phi_1, \phi_2)(0, \lambda)=\begin{vmatrix} \phi_1(0, \lambda) & \phi_2(0, \lambda) \\ \phi_1'(0, \lambda) & \phi_2'(0, \lambda) \end{vmatrix}=1.$$

Since $\phi_1(t+\omega, \lambda)$ and $\phi_2(t+\omega, \lambda)$ are also linearly independent solutions of the SL problem, they can be expressed in the form

$$\begin{aligned} \phi_1(t+\omega, \lambda)&=\alpha_{11}\phi_1(t, \lambda)+\alpha_{12}\phi_2(t, \lambda), \\ \phi_2(t+\omega, \lambda)&=\alpha_{21}\phi_1(t, \lambda)+\alpha_{22}\phi_2(t, \lambda), \end{aligned} \tag{2.3}$$

where α_{11}, α_{12}, α_{21} and α_{22} are constants. Using the initial conditions given in Eq. (2.2), it is easy to verify that

$$\alpha_{i1}=\phi_i(\omega, \lambda), \qquad \alpha_{i2}=\phi_i'(\omega, \lambda), \qquad i=1, 2. \tag{2.4}$$

Then the function $\psi(t, \lambda)$ in Eq. (2.1) can be expressed as

$$\psi(t, \lambda)=c_1\phi_1(t, \lambda)+c_2\phi_2(t, \lambda), \tag{2.5}$$

where either c_1 or c_2 can be zero, but not both zero. Substituting Eqs. (2.3) and (2.5) into Eq. (2.1) gives

$$\begin{aligned} &((\alpha_{11}-\rho)\,c_1+\alpha_{21}c_2)\,\phi_1(t, \lambda) \\ &\quad+(\alpha_{12}c_1+(\alpha_{22}-\rho)\,c_2)\,\phi_2(t, \lambda)=0. \end{aligned}$$

Hence

$$\begin{aligned} (\alpha_{11}-\rho)\,c_1+\alpha_{21}c_2&=0, \\ \alpha_{12}c_1+(\alpha_{22}-\rho)\,c_2&=0. \end{aligned} \tag{2.6}$$

Consequently the condition for a non-trivial solution is

$$\rho^2-(\alpha_{11}+\alpha_{22})\,\rho+(\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21})=0. \tag{2.7}$$

By a basic result about the Wronskian [6], we have

$$\begin{aligned} \alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}&=W(\phi_1, \phi_2)(\omega, \lambda) \\ &=W(\phi_1, \phi_2)(0, \lambda)\,e^{-\int_0^\omega (p'/p)\,dt} \\ &=1. \end{aligned}$$

Thus Eq. (2.7) becomes $\rho^2-(\alpha_{11}+\alpha_{22})\,\rho+1=0$.

Now denote

$$D(\omega, \lambda)=\alpha_{11}+\alpha_{22}=\phi_1(\omega, \lambda)+\phi_2'(\omega, \lambda). \tag{2.8}$$

It can be shown that the necessary and sufficient conditions for $\rho=1$ is $D(\omega, \lambda)=2$. Therefore the solutions of

$$D(\omega, \lambda)-2=0 \tag{2.9}$$

are identical to the eigenvalues of the SL problem with periodic boundary conditions (1.2).

If λ is a double root corresponding to two linearly independent eigenfunctions, then there exist two linearly independent solutions $(c_1, c_2)^T$ of Eq. (2.6) such that

$$\alpha_{12}=\alpha_{21}=0. \tag{2.10}$$

This condition implies that $\alpha_{11}=\alpha_{22}=1$. Equation (2.10) can be served as a criterion to test whether λ is a double root or not. Once the solutions $(c_1, c_2)^T$ are found, the eigenfunctions $\psi(t, \lambda)$ can be computed according to Eq. (2.5).

Up to now our discussion has been focused on the SL problem with periodic boundary conditions (i.e., $\rho=1$ in Eq. (2.1)). If μ is an eigenvalue for the SL problem with semi-periodic boundary conditions, however, the same argument previously presented will lead to the equation

$$D(\omega, \mu)+2=0 \tag{2.11}$$

and, correspondingly, $\rho=-1$ in Eq. (2.1). Equation (2.10) is still valid for testing a double root for μ.

The original SL eigenvalue problem with periodic or semi-periodic boundary conditions can now be cast into an initial value problem (IVP). The following iterative algorithm based on a shooting procedure in conjunction with Newton's method is then applied to solve the resulting IVP, and hence gives the solution of the SL eigenvalue problem.

Suppose λ is an eigenvalue of the SL problem with periodic boundary condition, and let $\lambda^{(k)}$ denote the kth approximation to λ. The choice for the initial value $\lambda^{(0)}$ will be discussed in the next section. Then, for $k = 0, 1, 2, \ldots$, until $|\lambda^{(k+1)} - \lambda^{(k)}| \leqslant \varepsilon$,

Step 1. Integrate numerically the system of the first-order ODEs,

$$\begin{aligned} y_1' &= y_2, \\ (py_2)' &= -(\lambda^{(k)}s - q)\, y_1, \\ y_{1,\lambda}' &= y_{2,\lambda}, \\ (py_{2,\lambda})' &= -(\lambda^{(k)}s - q)\, y_{1,\lambda} - sy_1, \end{aligned} \tag{2.12}$$

under the initial conditions

$$y_1(0) = 1, \qquad y_2(0) = 0, \qquad y_{1,\lambda}(0) = 0, \qquad y_{2,\lambda}(0) = 0.$$

Where $y_1 = y$ and $y_{1,\lambda} = \partial y / \partial \lambda$. Note that there are many robust and efficient subroutines (such as the DO2NDF in NAG) for integrating the IVPs. Let the solution be

$$\begin{aligned} y_1(\omega, \lambda^{(k)}) &= c_1, \qquad & y_2(\omega, \lambda^{(k)}) &= c_2, \\ y_{1,\lambda}(\omega, \lambda^{(k)}) &= c_3, \qquad & y_{2,\lambda}(\omega, \lambda^{(k)}) &= c_4. \end{aligned} \tag{2.13}$$

Step 2. Integrate the system of ODEs (2.12) subject to the initial conditions,

$$y_1(0) = 0, \qquad y_2(0) = 1, \qquad y_{1,\lambda}(0) = 0, \qquad y_{2,\lambda}(0) = 0,$$

and let the corresponding solutions be

$$\begin{aligned} y_1(\omega, \lambda^{(k)}) &= d_1, \qquad & y_2(\omega, \lambda^{(k)}) &= d_2, \\ y_{1,\lambda}(\omega, \lambda^{(k)}) &= d_3, \qquad & y_{2,\lambda}(\omega, \lambda^{(k)}) &= d_4. \end{aligned} \tag{2.14}$$

Step 3. Update the $(k+1)$th approximation of the eigenvalue λ by the application of Newton's formula

$$\begin{aligned} \lambda^{(k+1)} &= \lambda^{(k)} - \frac{D(\omega, \lambda^{(k)}) - 2}{D'(\omega, \lambda^{(k)})} \\ &= \lambda^{(k)} - \frac{c_1 + d_2 - 2}{c_3 + d_4}. \end{aligned} \tag{2.15}$$

It is worthwhile to mention that, instead of using the formula explicitly stated in Eq. (2.15), a subroutine such as the NEQNJ, which is an IMSL routine based on a modified M. J. D. Powell's hybrid algorithm, can be used to solve the nonlinear equation. Note that the NEQNJ subroutine is essentially a variation of Newton's method.

The iterative process is repeated until the difference between two (or three) successive approximations $\lambda^{(k+1)}$ and $\lambda^{(k)}$ is less than a prescribed tolerance ε, and $\lambda^{(k+1)}$ is then accepted as an eigenvalue for the SL problem. If

$$\max\left(\frac{|c_2| + |d_1|}{|c_1|}, \frac{|c_2| + |d_1|}{|d_2|} \right) < \delta,$$

where $\delta > 0$ is a given small constant, $\lambda^{(k)}$ and $\lambda^{(k+1)}$ are regarded as a double root.

The above iterative procedure can be applied in a straightforward manner for $\mu^{(k)}$, the eigenvalue of the SL problem with semi-periodic boundary conditions. The modification required in the above iterative procedure is replacing λ by μ and the equation $D(\omega, \lambda^{(k)}) - 2 = 0$ by $D(\omega, \mu^{(k)}) + 2 = 0$.

3. CONVERGENCE AND CHOICE OF STARTING VALUES

Suppose the numerical errors due to integrating IVPs are small and can be neglected. Since the iterative sequences $\{\lambda^{(k)}\}$ or $\{\mu^{(k)}\}$ are obtained by the application of Newton-type iteration, a sufficient condition of convergence and the rate of convergence of the iterative procedure presented in the last section then follow from Kantorovich's results [13].

Consider

$$D(\omega, \lambda) \pm 2 = 0;$$

i.e.,

$$\phi_1(\omega, \lambda) + \phi_2'(\omega, \lambda) \pm 2 = 0.$$

Note that $\phi_1(\omega, \lambda)$ and $\phi_2'(\omega, \lambda)$ are analytical functions of λ, and so is the function $D(\omega, \lambda)$.

THEOREM 1 (cf. [13]). *Let $\lambda^{(0)}$ be the starting approximation and let the subsequent approximations $\lambda^{(k)}$ be computed via Newton's formula in the shooting algorithm. Suppose the following conditions:*

(i) $|[D_\lambda(\omega, \lambda^{(0)})]^{-1}| \leqslant B$,

(ii) $|D(\omega, \lambda^{(0)})| \leqslant \eta - 2$, *where* $\eta \geqslant 2$,

(iii) $|D_{\lambda\lambda}(\omega, \lambda)| \leqslant \kappa$, *for* $\lambda \in \delta(\lambda^{(0)}, r)$, *where r is defined below,*

are satisfied. Now, if

$$\gamma = \kappa B^2 \eta \leqslant \tfrac{1}{2}$$

and

$$r \geqslant r_0 = \frac{1-\sqrt{1-2\gamma}}{\gamma} B\eta,$$

then the sequence $\{\lambda^{(k)}\}$ *or* $\{\mu^{(k)}\}$ *converges to an eigenvalue* λ^* *or* μ^* *of the SL problem with periodic or semi-periodic boundary conditions, such that*

$$|\lambda^* - \lambda^{(0)}| \leqslant r_0$$

or

$$|\mu^* - \mu^{(0)}| \leqslant r_0.$$

Moreover, the rate of convergence is characterized by the inequality

$$|\lambda^* - \lambda^{(k)}| \leqslant \frac{1}{2^k} (2\gamma)^{2^k} \frac{\eta}{\gamma}, \qquad k = 0, 1, ...,$$

or

$$|\mu^* - \mu^{(k)}| \leqslant \frac{1}{2^k} (2\gamma)^{2^k} \frac{\eta}{\gamma}, \qquad k = 0, 1, ...$$

The starting eigenvalues $\lambda^{(0)}$ and $\mu^{(0)}$ play an important role in the success of the shooting algorithm. We now examine how to choose $\lambda^{(0)}$ and $\mu^{(0)}$ for the iterative procedure presented in the last section.

Let $\{\lambda_i\}$ and $\{\mu_i\}$ be the eigenvalues of the SL problem with periodic and semi-periodic boundary conditions, respectively. Although Eq. (1.3) reveals the distribution of the eigenvalues, more useful information could be obtained from the following theorem.

THEOREM 2 [9]. *Let* $D(\lambda) := D(\omega, \lambda)$ *be defined as in Section* 2, *then*

(i) $D(\lambda) > 2$ *in the intervals* $(-\infty, \lambda_0)$ *and* $(\lambda_{2m+1}, \lambda_{2m+2})$,

(ii) $D(\lambda)$ *decreases from* 2 *to* -2 *in the intervals* $[\lambda_{2m}, \mu_{2m}]$,

(iii) $D(\lambda) < -2$ *in the intervals* (μ_{2m}, μ_{2m+1}),

(iv) $D(\lambda)$ *increases from* -2 *to* 2 *in the intervals* $[\mu_{2m+1}, \lambda_{2m+1}]$.

A typical graph of $D(\lambda)$ is plotted in Fig. 1. Assume that the eigenvalues are ordered as $\lambda_0 < \mu_0 \leqslant \mu_1 < \lambda_1 \leqslant \lambda_2 < \cdots$. We now present a strategy to compute the starting values for $\lambda_i^{(0)}, \mu_i^{(0)}, i = 0, 1, 2, ...$ Suppose $-N$, the lowest bound of the eigenvalues, can be determined such that $-N < \lambda_0$. Let m be the index associated with the eigenvalue; the starting eigenvalues can then be estimated according to the following order: $\lambda_0^{(0)}, \mu_0^{(0)}, \mu_1^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, ...$ Let $h_{j,m}$ denote the step size which depends on m, and the subscript j that could be either 0, 1, or 2. When $\lambda_0^{(0)}$ is being sought, $h_{0,m}$ is used; $h_{1,m}$ is used in $[\lambda_{2m}^{(0)}, \mu_{2m}^{(0)}]$ or $[\mu_{2m+1}^{(0)}, \lambda_{2m+1}^{(0)}]$ when

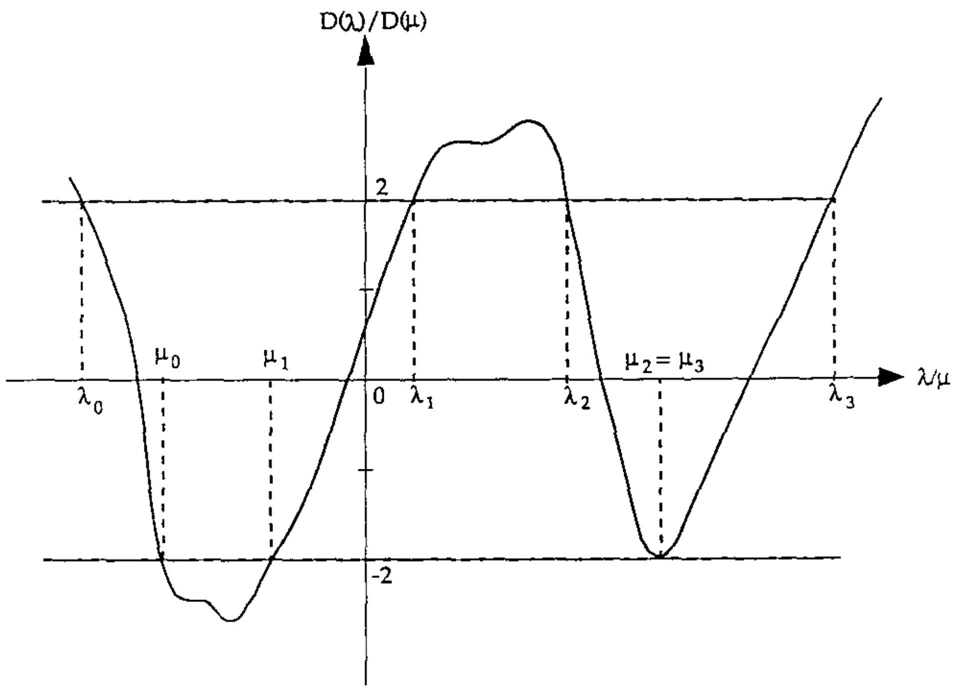

FIG. 1. Eigenvalue distributions.

the starting eigenvalues $\mu_{2m}^{(0)}$ or $\lambda_{2m+1}^{(0)}$ are sought; and $h_{2,m}$ is used in $(\lambda_{2m+1}^{(0)}, \lambda_{2m+2}^{(0)})$ or $(\mu_{2m}^{(0)}, \mu_{2m+1}^{(0)})$ for $\lambda_{2m+2}^{(0)}$ or $\mu_{2m+1}^{(0)}$. Now define

$$x_i^{(k)} = x_i^{(0)} + kh_{j,m}, \tag{3.1}$$

where $k = 1, 2, \ldots$. Assuming that the values of $x_i^{(0)}$ and $h_{j,m}$ are given, the values of $D(x_i^{(k)})$ can then be computed. Suppose when $k = k^*$,

$$|D(x_i^{(k^*)}) - 2| \leqslant \delta, \tag{3.2}$$

where δ is a prescribed small positive constant, then $x_i^{(k^*)}$ is taken as the starting value as $\lambda_i^{(0)}$. It is worth noting that when searching for the starting value $\lambda_i^{(0)}$ for $i = 2m+1$, it may be possible that Eq. (3.2) is not satisfied but that

$$D(x_i^{(k^*+1)}) \leqslant D(x_i^{(k^*)}). \tag{3.3}$$

In this case, λ_i and λ_{i+1} can be regarded as a double root and the starting values $\lambda_i^{(0)}$ and $\lambda_{i+1}^{(0)}$ are set to be

$$\lambda_i^{(0)} = \lambda_{i+1}^{(0)} = \tfrac{1}{2}(x_i^{(k^*+1)} + x_i^{(k^*)}).$$

The initial value of $x_i^{(0)}$ in Eq. (3.1) can be chosen as follows. Suppose the first eigenvalue λ_0 is being sought, then a natural choice for $x_i^{(0)}$ is $-N$. Hence, Eq. (3.1) becomes $x_i^{(k)} = -N + kh_{0,m}$. Once $\lambda_0^{(0)}$ is computed, $x_i^{(0)} = \lambda_0^{(0)}$ could be used when the starting value $\mu_0^{(0)}$ is being considered. Having determined $\mu_0^{(0)}$, the values for $\mu_1^{(0)}$ can be computed in a similar manner, and so on.

The strategy for the starting values presented in Eqs. (3.1) and (3.2) requires the choices of δ and $h_{j,m}$, where $j = 0$, 1, or 2. In practice, one often does not know the best values for δ and $h_{j,m}$, since it will vary from problem to problem, depending upon the distribution of the eigenvalues. In this paper, we suggest as a general guideline that δ is taken in the range from 0.1 to 0.5, $h_{0,m}$ is a constant between 5 and 10, $h_{1,m} = m^2/2$ and $h_{2,m} = m/2$, where m is the index of the eigenvalue.

4. NUMERICAL IMPLEMENTATION AND COMPUTED RESULTS

The convergence analysis presented in the previous section does not take into account the numerical errors introduced in the integration process for solving the IVPs. It is well known that for the SL problem the eigenfunction associated with the eigenvalue λ_m oscillates. Moreover, for large m, the corresponding eigenfunction is highly oscillatory. Consequently, the IVPs may be very ill-conditioned, resulting in inaccurately computed eigenvalues when m is large.

In order to overcome this difficulty, the following scaling procedure is implemented in Steps 1 and 2 of the iterative process presented in Section 2. For simplicity, suppose $p(t) = 1$ in Eq. (1.1) and denote the function

$$z(t) = \frac{1}{c}\, y(t), \tag{4.1}$$

in which the scaling factor c is defined as

$$c = [\max_{0 \leqslant t \leqslant \omega} |\lambda s(t) - q(t)|]^{1/2}. \tag{4.2}$$

Now instead of Eq. (2.12) used in Step 1 of the iterative algorithm, we integrate the system of ODEs,

$$\begin{aligned} z_1' &= cz_2, \\ z_2' &= -\frac{1}{c}(\lambda^{(k)}s - q)\, z_1, \\ z_{1,\lambda}' &= cz_{2,\lambda}, \\ z_{2,\lambda}' &= -\frac{1}{c}(\lambda^{(k)}s - q)\, z_{1,\lambda} - \frac{1}{c} s z_1. \end{aligned} \tag{4.3}$$

The corresponding initial conditions are replaced by

$$z_1(0) = \frac{1}{c}, \qquad z_2(0) = 0, \qquad z_{1,\lambda}(0) = 0, \qquad z_{2,\lambda}(0) = 0.$$

Similarly, the scaled system (4.3) is used in Step 2, subject to the initial conditions

$$z_1(0) = 0, \qquad z_2(0) = \frac{1}{c}, \qquad z_{1,\lambda}(0) = 0, \qquad z_{2,\lambda}(0) = 0.$$

We now present computational results to illustrate the effectiveness of the shooting algorithm developed in Section 2 for the Sturm–Liouville eigenvalue problems with periodic and semi-periodic boundary conditions. Let us consider the following three different kinds of equations.

Problem 1. $-y'' = \lambda y$, $-1 \leqslant t \leqslant 1$. The eigenvalues corresponding to periodic boundary conditions are given by

$$\lambda_0 = 0, \qquad \lambda_{2m+1} = \lambda_{2m+2} = (m+1)^2 \pi^2, \qquad \text{for} \quad m \geqslant 0.$$

The eigenvalues corresponding to the semi-periodic boundary conditions are

$$\mu_{2m} = \mu_{2m+1} = (2m+1)^2 \pi^2/4, \qquad \text{for} \quad m \geqslant 0.$$

Hence, all eigenvalues λ_i and μ_i, except λ_0, are repeated.

TABLE I

Eigenvalues for Problem 1

m	$\lambda_m^{(0)}$	λ_m^*	λ_m	$\mu_m^{(0)}$	μ_m^*	μ_m
0	−0.50	0.000000	-1.733×10^{-15}	2.00	2.467402	2.467401
1	9.25	9.869606	9.869608	2.75	2.467402	2.467400
2	10.00	9.869606	9.869610	21.00	22.206615	22.206608
3	38.50	39.478426	39.478426	22.50	22.206615	22.206609
4	40.00	39.478426	39.478390	60.50	61.685040	61.685015
5	86.25	88.826458	88.826483	62.75	61.685040	61.685076
6	90.00	88.826458	88.826484	118.00	120.902679	120.902748
7	155.00	157.913703	157.913544	121.00	120.902679	120.902787
8	160.00	157.913703	157.913712	197.00	199.859531	199.859674
9	242.75	246.740161	246.739903	200.75	199.859531	199.859442
10	249.00	246.740161	246.740116			

Problem 2. $-y'' = \lambda s(t)\, y$, $-1 \leqslant t \leqslant 1$, where

$$s(t) = \begin{cases} 1, & -1 \leqslant t < 0, \\ 9, & 0 \leqslant t < 1. \end{cases}$$

The exact solutions for this problem can be found in [9]. The eigenvalues corresponding to the periodic boundary conditions are

$$\lambda_0 = 0,$$
$$\lambda_{4m+1} = [(2m+1)\,\pi - \alpha]^2/4,$$
$$\lambda_{4m+2} = [(2m+1)\,\pi + \alpha]^2/4,$$
$$\lambda_{4m+3} = \lambda_{4m+4} = (m+1)^2\,\pi^2,$$

where $m \geqslant 0$, $\alpha = \cos^{-1}(\frac{7}{8})$, and $0 < \alpha < \pi/2$. For the semi-periodic boundary conditions, the eigenvalues μ_i are

$$\mu_{4m} = (m\pi + \tfrac{1}{2}\beta)^2,$$
$$\mu_{4m+1} = (m\pi + \tfrac{1}{2}\gamma)^2,$$
$$\mu_{4m+2} = [(m+1)\,\pi - \tfrac{1}{2}\gamma]^2,$$
$$\mu_{4m+3} = [(m+1)\,\pi - \tfrac{1}{2}\beta]^2,$$

TABLE II

Eigenvalues for Problem 2

m	$\lambda_m^{(0)}$	λ_m^*	λ_m	$\mu_m^{(0)}$	μ_m^*	μ_m
0	−2.50	0.000000	-1.141×10^{-15}	0.50	0.322430	0.322430
1	2.00	1.737430	1.737424	1.00	0.875964	0.875964
2	3.00	3.325067	3.325067	5.00	4.864952	4.864950
3	9.00	9.869606	9.869526	7.00	6.624260	6.624257
4	11.00	9.869606	9.869670	14.00	13.759812	13.759813
5	20.00	19.889006	19.889041	17.00	16.626188	16.626185
6	24.50	24.651917	24.651850	28.50	28.593154	28.593149
7	38.50	39.478426	39.478829	32.50	32.665304	32.665287
8	40.50	39.478426	39.477954	48.00	46.936408	46.936409
9	58.00	57.779795	57.779888	53.00	52.115625	52.115608
10	65.50	65.717980	65.717887			

where $m \geqslant 0$, $\beta = \cos^{-1}((1+\sqrt{33})/16)$, $\gamma = \cos^{-1}((1-\sqrt{33})/16)$, and $0 < \beta < \gamma < \pi$. Therefore, some eigenvalues λ_i are distinct and some λ_i are repeated. The eigenvalues corresponding to the semi-periodic boundary conditions, however, are all distinct.

Problem 3. The Mathieu equation

$$y'' + (\lambda - 2q \cos 2t)\, y = 0, \qquad -\pi/2 \leqslant t \leqslant \pi/2.$$

The parameter q is set to be five for Problem 3. The eigenvalues corresponding to both periodic and semi-periodic boundary conditions are all distinct. However, the value of λ_{2m+1} or μ_{2m} is very close to that of λ_{2m+2} or μ_{2m+1} as m increases (cf. [18]).

The computational results for Problems 1, 2, and 3 are summarized in the Tables I, II, and III, respectively. The strategy for the starting eigenvalues presented in Section 3 is applied to compute $\lambda_m^{(0)}$ and $\mu_m^{(0)}$. It should be pointed out that the suggested values for δ and $h_{j,m}$ are used, except that $h_{2,m}$ is set to be a constant unity for Problem 3. In the tables, λ_m and μ_m denote the computed eigenvalues, and λ_m^* and μ_m^* are the exact eigenvalues corrected to six decimal places. The calculation was carried out in single precision on a CYBER 860 computer. The tolerance ε in the iterative algorithm is 10^{-4}.

TABLE III

Eigenvalues for Problem 3

m	$\lambda_m^{(0)}$	λ_m^*	λ_m	$\mu_m^{(0)}$	μ_m^*	μ_m
0	−7.50	−5.800046	−5.800046	−5.50	−5.790081	−5.790081
1	2.50	2.099460	2.099460	−2.50	1.858188	1.858188
2	7.50	7.449110	7.449110	9.50	9.236328	9.236327
3	15.50	16.648220	16.648212	11.50	11.548832	11.548832
4	17.50	17.096582	17.096582	24.00	25.510816	25.510710
5	34.50	36.358867	36.358743	27.00	25.549972	25.550027
6	37.50	36.360900	36.361011	46.50	49.261383	49.261446
7	61.50	64.198841	64.198831	50.50	49.261455	49.261424
8	65.50	64.198842	64.198816	78.00	81.156455	81.156485
9	96.50	100.126369	100.126385	82.00	81.156455	81.156435
10	101.50	100.126369	100.126355			

5. CONCLUSIONS

A new shooting algorithm has been developed for the Sturm–Liouville eigenvalue problems with periodic and semi-periodic boundary conditions. From the numerical results reported in the last section, we conclude that the present technique is efficient and that the corresponding eigenvalues can be accurately computed. However, it should be pointed out that the good performance of this algorithm depends upon the choice of the starting values. Although a general guideline is given for the starting values, further work is needed so that the technique could be applied even when information on the eigenvalue distribution is not given.

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